DENSITY-WAKE OF A CHARGED PARTICLE MOVING THROUGH A PLASMA

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ABSTRACT. The method developed by the author in a previous paper has been applied to investigate the density wake created by a moving test particle in the plasma. These wakes are analysed separately for different harmonics of the cyclotron frequency, and for small velocity perpendicular to the applied magnetic field. It is shown that only the first few harmonics are important to develop the wake in the plasma.

INTRODUCTION

In a previous paper by the author, (Majumdar, 1963), the shape of the wake of a test particle moving through plasma placed in a steady magnetic field has been investigated. In the present paper, we extend the calculation for a general motion of the test particle in the plasma.

It is known that when we neglect the motion of plasma ions, there are three different wave motions in the plasma : one is a plasma-electron wave the two others are transverse e.m. waves as shown by Oster (1960). One of the e.m. waves is strongly coupled with the plasma wave, which is a slow wave, while the other e.m. wave propagates freely and is a fast wave. (Allis, *et al.* 1962). In this paper we shall investigate only qualitatively, the nature of the wake of the test particle on the above assumptions of two coupled slow waves and a fast wave. We shall employ the same set of hydrodynamic equations as has been done in (Majumdar, 1962). It has been shown there that the wake of the moving charged particle is determined solely by the nature of the wave-surface surrounding it. This wave surface is obtained by plotting the parallel versus perpendicular (to the external magnetic field) component of the wave-vector **k**. The actual wake of the moving particle, i.e. the disturbance in the plasma created by it in the form of density wave, is obtained by taking the polar reciprocal of the wave surface. If the wave surface is real, the charge density takes the form of a radiated wave emanating from the particle. If it is imaginary, this radiated wave is to be replaced by a damped density distribution around the particle.

BASIC EQUATIONS

The motion of the plasma electrons is represented by the linearised transport equation (Spitzer, 1956).

$$\frac{\partial \boldsymbol{v}}{\partial t}+\frac{e}{m}\boldsymbol{E}+\frac{V^2}{n_0}\nabla n+\omega_c \boldsymbol{v}\times \boldsymbol{e}_z=0, \qquad \text{... (1)}$$

together with the equation of continuity

$$\frac{\partial n}{\partial t}+\nabla\cdot(n_0\boldsymbol{v})=0, \qquad \text{... (2)}$$

and the set of Maxwell's equations :

$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \qquad \text{... (3)}$$

$$\nabla\times\boldsymbol{H}=\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}-\frac{4\pi}{c}n_0e\boldsymbol{v}+\frac{4\pi q}{c}\boldsymbol{V}_0(t)\,\delta(\boldsymbol{r}-\boldsymbol{r}_T). \qquad \text{... (4)}$$

In these four equations, $\boldsymbol{v}$ and n are perturbations in velocity and density of plasma electrons, n_0 and V are the average density and average velocity, and

$$\omega_c=\frac{eH_0}{mc}$$

is the cyclotron frequency, H_0 being the external magnetic field applied in ;the Z-direction, represented by the unit vector $\boldsymbol{e}_z$. q is the test particle's charge moving with a velocity $\boldsymbol{V}_0(t)$, $\boldsymbol{r}_T$ is its position denoted by the Dirac delta-function δ.

We take Fourier transform w.r.t. space and time of the form

$$f(\boldsymbol{k},\omega)=\int\int_{-\infty}^{+\infty} d\boldsymbol{r}\,dt\,f(\boldsymbol{r},t)e^{i(\mathbf{k}\cdot\boldsymbol{r}-\omega t)},$$

of all the variables in eqn.(1) to (4), and then eliminate the quantities $\boldsymbol{v}$, $\boldsymbol{H}$ and n. This gives us the following equation for the electric field in the plasma :

$$a\boldsymbol{E}+\boldsymbol{k}\left(i\frac{V^2}{\omega}4\pi q\boldsymbol{k}\cdot\Gamma-[c^2-V^2]\,\boldsymbol{k}\cdot\boldsymbol{E}\right)-ib\,(\boldsymbol{E}\times\boldsymbol{e}_z)$$

$$-i\frac{\omega_c}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E})(\boldsymbol{k}\times e_z)+4\pi q\omega_c\Gamma\times\boldsymbol{e}_z-i4\pi q\omega\Gamma=0, \qquad \text{... (5)}$$

where

$$\left.\begin{aligned} a&=k^2c^2+\omega_p^2-\omega^2\\ b&=\omega_c\left(\omega-\frac{c^2k^2}{\omega}\right)\\ \text{and}\quad \omega_p&=\left(\frac{4\pi n_0e^2}{m}\right)^{\frac{1}{2}}\end{aligned}\right\} \qquad \text{... (6)}$$

is the plasma frequency. The quantity Γ is the source function generated by the moving test particle

$$\Gamma = \int_{-\infty}^{+\infty} V_0(t) e^{-i\mathbf{k}\cdot\mathbf{r}_T + i\omega t}\, dt. \quad \ldots \quad (7)$$

In eqn. (5) $\boldsymbol{E}$ is actually the fourier transform $\boldsymbol{E}(\mathbf{k}, \omega)$ of the electric field. Now, our aim is to obtain an expression for the charge density in the plasma, caused by the moving test particle. For this, we take Fourier transform of Poisson's equation

$$\vec{\nabla}\cdot\boldsymbol{E} = -4\pi ne + 4\pi q\delta(\boldsymbol{r}-\boldsymbol{r}_T), \quad \ldots \quad (8)$$

and combine the transform equation with eqn. (5). We thus obtain finally the following expression for the Fourier transform of the charge density :

$$\begin{aligned}\rho(\boldsymbol{k}, \omega) &= en(\boldsymbol{k}, \omega) = \\ &= qI_0\left(1 + k_z V_{0z}\frac{P}{D}\right) \\ &+ qI_{+\Omega} e^{i(\phi-\theta)} k_\rho V_{0\rho}\frac{Q + a\omega_c\omega_p^2}{2D} \\ &+ qI_{-\Omega} e^{-i(\phi-\theta)} k_\rho V_{0\rho}\frac{Q - a\omega_c\omega_p^2}{2D}, \quad \ldots \quad (9)\end{aligned}$$

where
$$I_0 = \int_{-\infty}^{+\infty} e^{-i\mathbf{k}\cdot\mathbf{r}_T + i\omega t}\, dt, \quad \ldots \quad (10a)$$

$$I_{\pm\Omega} = \int_{-\infty}^{+\infty} e^{-ik\cdot r_T + i(\omega\pm\Omega)t}\, dt, \quad \ldots \quad (10b)$$

and
$$D = a(a^2 - b^2) - (a^2k^2 - b^2k_z^2)(c^2 - V^2) - ab\frac{\omega_c}{\omega}c^2k_\rho^2, \quad \ldots \quad (11a)$$

$$P = \omega(a^2 - b^2) - \frac{V^2}{\omega}(a^2k^2 - b^2k_z^2), \quad \ldots \quad (11b)$$

$$Q = P + \omega_c\omega_p^2 b. \quad \ldots \quad (11c)$$

In eqn. (9), the subscripts ρ and z denote the component perpendicular and parallel to z-axis,

$$\left.\begin{aligned} k^2 &= k_\rho^2 + k_z^2;\ k_\rho^2 = k_x^2 + k_y^2 \\ V_0^2 &= V_{0\rho}^2 + V_{0z}^2;\ V_{0\rho}^2 = V_{0x}^2 + V_{0y}^2 \end{aligned}\right\} \quad \ldots \quad (12)$$

$$\left.\begin{aligned} k_x &= k_\rho \cos\theta;\ k_y = k_\rho \sin\theta \\ V_{0x} &= V_{0\rho} \cos(\Omega t+\phi);\ V_{0y} = V_{0\rho} \sin(\Omega t+\phi) \end{aligned}\right\} \quad \cdots \quad (13)$$

and finally

$$\Omega = \frac{qH_0}{Mc} \quad \cdots \quad (14)$$

is the cyclotron frequency of the test particle.

APPROXIMATIONS FOR LOWER HARMONICS OF CYCLOTRON FREQUENCY

Without any loss of generality we assume that $k_y = 0$ and $\phi = 0$. Let us also identify the moving test particle with the plasma electrons, so that $\Omega = \omega_c$. In that case, we can write

$$e^{-i\vec{k}\cdot\vec{r}_T} = e^{-iz\sin\omega_c t - ik_z V_{0z}t},$$

where

$$z = \frac{k_x V_{0\rho}}{\omega_c} \quad \cdots \quad (15)$$

Using the well known expansion

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{+\infty} J_n(z) e^{in\theta}$$

we then obtain from eqns. (10),

$$I_0 = \sum_{n=-\infty}^{+\infty} J_n(z)\delta(\omega - n\omega_c - k_z V_{0z})$$

$$I_{\pm\Omega} = \sum_{n=-\infty}^{+\infty} J_n(z)\,\delta(\omega - n\omega_c \pm \omega_c - k_z V_{0z}).$$

Using these results in eqn. (9) and taking the inverse transform w.r.t. time, we get the following expression for ρ:

$$\rho(\mathbf{k}) = e\sum_{n=-\infty}^{+\infty} e^{i(n\omega_c + k_z V_{0z})t} J_n(z)\left(1 + k_z V_{0z}\frac{P}{D}\right)_{\omega = n\omega_c + k_z V_{0z}}$$

$$+ce^{-i\theta}\sum_{-\infty}^{+\infty} e^{i[(n-1)\omega_c + k_z V_{0z}]t} J_n(z) k_\rho V_{0\rho}\left(\frac{Q + a\omega_c\omega_p^2}{2D}\right)_{\omega = (n-1)\omega_c + k_z V_{0z}}$$

$$+ee^{+i\theta}\sum_{-\infty}^{+\infty} e^{i[(n+1)\theta + k_z V_{0z}]t} J_n(z) k_\rho V_{0\rho}\left(\frac{Q - a\omega_c\omega_p^2}{2D}\right)_{\omega = (n+1)\omega_c + k_z V_{0z}} \quad \cdots \quad (16)$$

Now, for small values of V_0 and large magnetic field, the quantity z given by (15) is much less than unity. In that case we need to consider only lower harmonic terms (small n) in the above summations over n, and for all practical purposes, it is sufficient to keep only $n = 0$ and $n = \pm 1$ terms. After obtaining the expressions for $\rho(\mathbf{k})$ for $n = 0$ and $n = \pm 1$, we can apply the method developed in (Majumdar, 1963) to integrate them over the variable $\mathbf{k}$ and obtain expressions for $\rho(\mathbf{r}, t)$ for $n = 0$ and $n = \pm 1$ terms. In the process of doing that integration it is observed that the wave-surfaces are given by plotting k_z vs. k_ρ from the dispersion relation

$$D = 0$$

This plot, as is shown in (Majumdar, 1963), will determine the wake of the moving test particle in plasma.

NATURE OF THE WAVE SURFACE

The dispersion relation (17)is a six degree equation and is quite complicated. We therefore make two simplifying assumption : (i) there are one fast wave and two coupled slow waves, and (ii) $k^2c^2 >> \omega_p^2$ and V^2, $V_z^2 << c^2$. The implications regarding these assumptions has been fully explained in (Majumdar, 1963). With these assumptions, eqn. (7) degenerates into two separate equations : For fast waves :

$$(k^2c^2-\omega^2)[2\omega_p^2\omega^2+\omega_c^2\omega^2-\omega_c^2c^2k_z^2]+\omega^2\omega_p^2(k^2V^2+\omega_p^2-\omega^2) = 0, \quad \ldots \quad (18)$$

and for slow waves :

$$k^2\omega^2(k^2V^2+\omega_p^2-\omega^2)-(k^2V^2+\omega_p^2)c^2k_z^2+\omega_c^2k^2\omega^2=0 \quad \ldots \quad (19)$$

As has alredy been shown in connection with eqn. (16), ω in eqns. (18) and (19) are given by

$$\omega = k_z V_{0z} \pm n\omega_c, \quad \ldots \quad (20)$$

where $n = 0, 1, 2$. For $n = 0$, eqns. (18)) and (19) reduce to eqns. (33) and (37) of (Majumdar, 1963), which is true only when the test particle is moving parallel to the magnetic field. Hence we come to the conclusion that for the fundamental frequency $\omega = k_z V_{0z}$, the effect of the test particle on its wake is the same as if the particle is moving parallel to the magnetic field.

To see the effects of the harmonics of cyclotron frequency, we put $V_{oz} = 0$ in (20) and set $\omega = \pm n\omega_c$, $(n = 1, 2)$ in equns. (18) and (19). Thus in the case of slow waves, eqn. (19) for $n = 1$ reduces to the following simple form :

$$k_\rho^2 \left(k_\rho^2+k_z^2+\frac{\omega_p^2}{V^2}\right) = 0. \quad \ldots \quad (21)$$

Now, it has been shown in (Majumdar, 1963) that a real wave surface means a radiation from the test particle, whereas an imaginary surface denotes an attenuated charge distribution arround the test particle. The surface represented by (21) is an imaginary one, so that for $\omega = \omega_c$, there is no real radiation from the moving test particle.

Setting now $n = 2$, in eqn. (19) we obtain the following equation :

$$k_\rho^4 + \frac{7}{4} k_\rho^2 k_z^2 + \frac{3}{4} k_z^4 + \frac{\omega_p^2 - 3\omega_c^2}{V^2} k_\rho^2 + \frac{3(\omega_p^2 - \omega_c^2)}{V^2} k_z^2 = 0. \quad \ldots \quad (22)$$

For $3\omega_p^2 \neq \omega_c^2$, eqn. (22) is a non-factorizable bi-quadratic equation. To get an idea of the wave surface we follow the procedure developed in (Majumdar, 1963). We first plot k_p^2 vs. k_z^2 from eqn. (22). This plot is always a hyperbola. From this plot, we take only that portion which lies in the first quadrant ($k_z^2 > 0, k_\rho^2 < 0$). Then taking square-root of each point of the curve which lies on this portion, we may separately plot k_ρ vs. k_z, and thus obtain the real wave surface. Thus, if

$$\alpha = -\frac{4}{V^2}(15\omega_p^2 - 3\omega_c^2)$$

$$\beta = +\frac{4}{V^2}(17\omega_p^2 - 3\omega_c^2), \qquad (23)$$

then a plot of k_z^2 vs. k_ρ^2 of eqn. (22), will be as shown in Fig. 1. It can be easily verified from Fig. 1, that

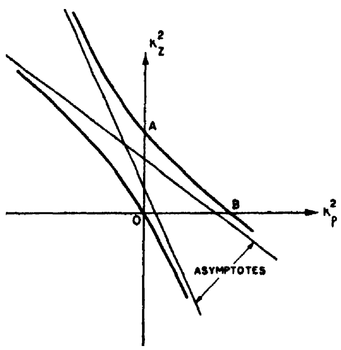

Fig. 1. k_ρ^2 vs. k_z^2 plot of eqn. (22).

$$OA = \frac{4}{V^2}(\quad - \omega_p^2); \; OB = \frac{1}{V^2}(3\omega_c^2 - \omega_p^2). \qquad (24)$$

Therefore, for real surface in k_ρ vs. k_z, plot, OA and OB should both be positive i.e., for radiative waves, we should have $\omega_c^2 > \omega_p^2$. Otherwise the wave surface is imaginary, and we have only attenuated charge distribution around the test particle. If we approximate the AB part of the curve (which lies in the first quadrant) of Fig. 1, by straight line AB, then its equation is given by

$$\frac{k_\rho^2}{OB} + \frac{k_z^2}{OA} \qquad (25)$$

This equation, in k_ρ vs. k_z plot, represent an ellipse if $OA \neq OB$, and a circle, if $OA = OB$, i.e. if $3\omega_p^2 = \omega_c^2$. In either case we have a radiated wave, either elliptical or, circular, radiating from the test particle. As explained before, when OA or OB has negative value, (i.e. $\omega_c^2 < \omega_p^2$), then this radiative wave is replaced by and attenuated (damped) charge distribution around the moving test particle. Once the wave surface around the test particle is obtained it is now a matter of geometrical construction to get the constant charge-density surface. Without going into the mathematical details (as has been done in (Majumdar, 1963), this latter surface can be obtained from the k-surface (wave-surface) by constructing to polar reciprocals. When the wave surface is real, its polar reciprocal is also real, meaning thereby a radiated wave from the test particle. For imaginary polar reciprocal of the k-surface, the radiated wave is replaced by a damped density distribution around the particle.

CONCLUSIONS

We have discussed the wave surface only for $n = 0$, 1 and 2. The calculation can be easily extended for higher values of n. For $n = 1, 2$, we had to assume that $V_{0z} = 0$. This has been necessary to simplify the mathematical complications. For non-zero V_{0z}, the wave surface is represented by a full-sixth degree equation which is very difficult to handle.

The method developed here can be applied to any kind of wave propagation in plasma (e.g., ion-cyclotron waves, Alfren waves etc.) for which it is only necessary to obtain the dispersion relation in a suitable form. The type of investigation developed here will give us the physical picture of how the charge density is associated with various kind of wave phenomena in magnetic plasma.

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